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Determination for the time-to-fracture of solids

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Abstract: A method to determine the time to fracture taking into account the physical mechanisms of microcracks and cracks formation is developed on the basis of the fractal model of fracture. The fractal dimension of a crack at different stages of its growth is determined theoretically. The damage evolution law which allows for the kinetic and microstructural properties of a material is obtained on the basis of the kinetic theory of strength. Conditions at which the microcracks accumulation gives way to the propagation of a large crack are determined with the use of the percolation theory. It is shown that the fractal dimension of the initial part of a crack is much more than the fractal dimension of the rest of crack.

1 Introduction

The time-to-fracture versus applied load relation is used often to determine the reliability and strength of products. In determining the relation, physical mechanisms of crack formation are seldom taken into account [1].

This paper seeks to determine the time-to-fracture versus stress relation taking into account the available data about the physical mechanisms of the macro- and microfracture, and using the kinetic theory of strength [1] and the fractal model of fracture.

The relation between the time-to-fracture and applied load (stress) was investigated by Zhurkov [1], Hsiao [2], Yokobori [3], L.Kachanov etc. Zhurkov has obtained experimentally the following formula [1]:

$$t = \psi \exp(-\gamma\sigma/k_B T) \tag{1}$$

where t= time-to-fracture, σ = applied stress, ψ and γ = kinetic constants of material, k_B= Boltzmann constant, T =temperature. This equation was explained as follows: fracture is caused by the accumulation of broken bonds between elements of the material; failure of the bonds is caused by the thermofluctuational process [2]. Such assumptions lead just to the equation like (1) [1,2]. One should note here that the formation of a microcrack is likely to be caused by the atomistic (i.e. thermo-activated, fluctuational) processes , but the formation of a large crack is known to be determined by the initiation and coalescence of microcracks, not by the accumulation of the broken atomistic bonds

[4,5]. T.Yokobori [3] used the analogy between fracture and the phase transition as well, and applied the methods of the statistical mechanics and the phase transition theory to determine the crack velocity.

It is known that the process of fracture consists on the several stages: the formation of damage (microcracks) in the random points of a loaded body; the coalescence of the microcracks and the formation of initial cracks; the growth of the initial cracks ; then, one of the cracks becomes largest and propagates with increasing velocity; the breakdown of the body or arrest of the crack [4-8]. Usually, the different theoretical methods are applied to model these stages: for example, the growth of a large crack can be modeled by the methods of the fracture mechanics; the damage evolution is studied with the use of the continuum damage mechanics [9], etc. Yet, to determine the time-to-fracture versus load relation taking into account the physical mechanism of the fracture, one should develop a model of the fracture which can include all these processes. Some investigations in this direction were conducted by Krajcinovic and Basista [10], Chelidze [11], Ostoja-Starzewski [12] etc. They applied the methods of the percolation theory to describe the formation of both damage (microcracks) and cracks. But the analogy between the percolation and crack formation is correct only when the crack is formed due to the coalescence of the microcracks, i.e. only for the initial crack and not for the large growing crack. When the crack grows, its velocity depends on its size [13,14], and it is not correct for any percolation cluster [17,18].

Here, the different stages of crack formation are analyzed on the basis of an unified approach, which includes the kinetic model of microcrack formation and a model of the formation of large cracks as a fractal cluster [15-18].

2 Analogy between the physical mechanisms of fracture and formation of different fractal objects

As mentioned above, the process of fracture of a non-cracked body consists on the following stages: initiation and formation of microcracks; coalescence of the microcracks and the formation of the initial cracks; autocatalytic propagation of the as-formed cracks, which can be arrested or can lead to the failure of a body [4,5,6]. The crack growth proceeds as follows [3-7]: under loading, one or more microcracks are formed in front of the crack tip. Then, a microcrack joins the crack and that leads to the crack growth. One can note that the microcrack accumulation, crack formation and the interaction between them differ essentially for the different stages of fracture: when there are no large cracks in the loaded body, the microcracks are formed randomly throughout the body and independently of one another. When their density exceeds some level they coalesce and form an initial crack. The more the microcracks density, the greater the probability of their coalescence and the formation of an initial crack. When the loaded body contains a crack (for example, if the initial crack has been already formed by the above described mechanism), the microcracks are formed not in the random points throughout the body, but just in the vicinity of a crack tip (it is determined by the stress concentration near the crack tip). In so doing, the density of the microcracks in the loaded body does not increase, since after the microcracks formation, they join the large crack [4,6].

It is of interest to compare the mechanisms of the formation of the initial and propagating crack with the two types of fractal objects: the fractal aggregates and the percolation clusters [15-18]. A percolation cluster is formed from randomly distributed stationary small elements and some minimal density of these elements is necessary for its formation. A fractal aggregate is formed due to the growth process (i.e. due to the random joining small elements to the aggregates). The difference between these two types of the fractal objects lies in the fact that the probability of the formation of a percolation cluster increases with increasing density of the elements. It is not true for the fractal aggregates: when a fractal aggregate grows the density of the elements can decrease or be constant [15,16]. One can see that the process of the formation of the initial crack from the microcracks can be considered as the percolation whereas the crack propagation can be taken as the formation of fractal aggregate. In this case the casual nature of the process of the microcrack formation in the vicinity of crack tip [5] is analogue of the casual nature of the joining particles in the growth of fractal aggregate (which is caused by the random movement of the particles being joined).

Thus, the properties of the initial and propagating crack are to be determined on the basis of different approaches: the percolation theory for the initial crack and the theory of growth of the fractal aggregates for the growing crack.

3 Fractal dimension of initial and growing cracks

Consider the fractal dimension of the initial and growing cracks. As shown above, the properties of the crack at these two stages of its formation are to be determined with the use of the different models of the fractal clusters . Determine the fractal dimension of an initial crack. Let us take some volume in which the crack can be considered as an infinite percolation cluster (i.e. the dimension of this volume is close to the presumed crack size). One can write the following formulas of the theory of critical phenomena [17,18]:

$$L \propto (x - R)^{-\nu} \tag{2}$$

$$N/L^d \propto (x - R)^{\beta} \tag{3}$$

where L - crack length (or, generally, a linear size of a percolation cluster), d - dimension of space, N - the amount of microcracks from which the macrocrack has been formed, R- the density of the microcracks in the volume (i.e. the damage parameter [9]), ν and β- the critical indices of the theory of critical phenomena. Taking into account the definition of the fractal dimension one can write:$D = d - \beta/\nu$ [17,18]. For two-dimmensional case, the values of β and ν are equal approximately [17]: $\beta = 0.139$, and $\nu = 1.33$. Substituting these values into the above formula, one can determine the fractal dimension of the initial macrocrack:$D = 1.89$.

The fractal dimension of the growing crack has been determined theoretically by Mishnaevsky Jr [7,8], and is about 1.3...1.5.

Thus, the fractal dimension of the initial part of the crack is much more than the fractal dimension of the rest part of crack. One can conclude that the surface roughness of the crack (which is the greater the greater the fractal dimension of the surface [15]) for initial crack (or when the crack is rather small) is much more than that for the large crack.

The fracture toughness is the more the greater the fractal dimension of the crack [19]. It follows herefrom that the specific energy needed to create the initial part of the crack is much more than that needed for the crack propagation.

4 Accumulation of microcracks and formation of initial crack

The initiation and accumulation of the microcracks (damage) in a non-cracked body can be described by the damage evolution law [9]. The damage evolution law is determined usually with the use of the thermodynamical model or experimental data [9]. Here, it is suggested to apply the kinetic theory of strength to do this.

It is supposed that the formation of microcracks (but not the crack formation) is caused just by the thermofluctuational processes [1,2] and can be described by the Eq.(1). Another version of Eq.(1) can be written thus [14,7]:

$$t = \psi_0 \exp(-\gamma_0 K_c / k_B T) \tag{4}$$

where K_c is the stress intensity factor in a point. Using Eq.(1) or (4) one can determine the damage evolution law. In the average, an unit increment of the damage parameter (i.e. the formation of a microcrack) occurs at a time interval, which is equal to t and depends on the applied stress. So, one can write

$$dR/dt = 1/t \tag{5}$$

where R- the damage parameter (here we consider for simplicity only the isotropic case), t - time. Taking into account the effective stress concept [9] and substituting the effective stress into Eqs.(5) and (1), we obtain

$$dR/dt = (1/\psi) \exp[\gamma\sigma/(1-R)] \tag{6}$$

Eq.(6) presents the damage evolution law which has been obtained on the basis of the thermofluctuational model of microfracture.

Consider a case when the microcrack is formed from a pileup of dislocations. It occurs when the stress at the point exceeds some critical level. The stress intensity factor in the vicinity of a pile-up of dislocations can be calculated as follows [4,7]:

$$K_c = \sqrt{\mu b n \sigma / (1-\nu_p)} \tag{7}$$

where μ- the shear modulus of the material, b - the Burgers vector of a dislocation, n - the number of the dislocations in a pile-up, ν_p- the Poisson's ratio.

Substituting Eq.(7) into (4), (5) we obtain the damage evolution law for metals which allows for the kinetic properties of the metal, and the distribution of dislocations:

$$dR/dt = (1/\psi_0)\ exp[\lambda\sqrt{\sigma/(1-R)}], \tag{8}$$

where $\lambda = \gamma_0 \sqrt{\mu b n}/(k_B T \sqrt{1-\nu_p})$.

Eq.(8) describes the damage evolution in metals with allowing for the physical mechanisms of the damage formation.

5 Condition of change in the regime of crack growth

Consider the conditions, at which the processes of the damage accumulation and the formation of initial crack due to the microcracks coalescence give way to the stable crack growth.

The initiation of the microcracks occurs in random points throughout the body. If the density of the microcracks in the given volume (which can be small or close to the volume of the loaded body) exceeds the percolation threshold, an infinite percolation cluster from the microcracks is formed. Such percolation cluster constitutes one of the initial cracks in the loaded body. So, the probability of the formation of a crack of length L is equal to the probability that xw/a^3 microcracks are formed in the volume $w = L^3$ (here, a is the average microcrack size). This probability can be determined as a probability of realizing this event (i.e. microcrack formation) xw/a^3 times in w/a^3 independent repeated trials, and described by the binomial law [20]:

$$p(k,i) = C_k^i R^i (1-R)^{k-i} \tag{9}$$

where $k = xw/a^3$, C- the binomial coefficient, p(k,i) is the probability of the formation of the cluster from i microcracks in the volume w. When the average density of the microcracks is small and w/a^3 ia rather large, this distribution law can be approximated by the the Poisson's law.

The linear size of a percolation cluster L and the number of elements from which the cluster has been formed are related as follows [7,18]:

$$i = (L/a)^D \tag{10}$$

where L- the linear size of a cluster , i -the number of elements which have formed this cluster (in this case, the number of microcracks), a - the size of an microcrack.

Substituting Eq(10) into the formula of the Poisson's distribution, one can obtain:

$$p(i) = [(Ri^{3/D})^i/i!]exp(-Ri^{3/D}) \tag{11}$$

where p(i) is the probability of the formation of a crack of i microcracks due to their coalescence, D- the fractal dimension of the crack.

Eqs.(10) and (11) describe the size distribution of the initial cracks which are formed in the loaded body due to the coalescence of the microcracks.

Thus, in the course of the microcracks coalescence, a number of the clusters of the microcracks (or initial cracks) are formed in the body. One of these clusters begins to grow autocatalytically due to the influence of its stress field on the intensity of the microcracks initiation in the vicinity of the crack tip, and the growth of the crack leads to the breakage of the body.

Consider the relation between the size of the maximal crack and the microcracks density in the loaded body.

The probability $p(i)$ is monotonically decreasing function of i. It is clear that a crack which grows with maximal velocity is of maximal size [4,7,13]. So, the probability $p(i_{max})$ for this crack is minimal. It means that there is only one such crack in the loaded body (a case of two cracks of the equal lengths can be neglected). So, one can write

$$p(i_{max})M = 1 \tag{12}$$

where $M = RV/(a^3 \sum_i p(i)i)$ - the number of the microcracks clusters in the loaded body, V - loaded body, i_{max} is the number of the microcracks which have formed the maximal initial crack.

Eqs.(11)-(12) describe the relation between the microcracks density (the damage parameter) in the loaded body and the size of maximal crack, which has been formed due to the microcracks coalescence.

Consider now the condition at which a cluster of the microcracks begins to grow intensively. It occurs provided that this cluster makes such stress concentration that the microcracks are formed not in the random points , but just in the vicinity of the crack (cluster) tip. To do this, the size of the cluster must exceed some critical value. So, this condition can be stated as follows: when the size of this cluster is smaller than critical one, the stress in a point may be determined with the use of the effective stress concept or strain equivalence principle [9]; when this cluster is greater than this critical value the stress field is determined by the stress concentration from the crack, and can be described by the formula of this kind [3]:

$$\sigma = \sigma_a \sqrt{L/2r} f(\theta) \tag{13}$$

where σ is a component of the stress tensor, σ_a=the normal stress acting on the specimen, r and θ are polar coordinates of the point of the microcrack initiation in the vicinity of the crack tip , $f(\theta)$ is a given function of θ [3,7]. If one supposes that the microcracks initiation is caused mainly by the shear stress, $f(\theta) = sin(\theta/2)cos(\theta/2)cos(3\theta/2)$ [3, 7].

The condition of changing the regime of the microcrack initiation from the random one to the regime which is determined by the stress field from the crack can be written in the following manner:

$$t_1 \leq t_2 \tag{14}$$

where t_1 and t_2 are time-to-formation of a microcrack in any point of the loaded body and in the vicinity of the crack tip, respectively. The value t_1 can be calculated with the use of Eqs. (5)and (8) as follows:

$$t_1 = \psi_0 \exp[-\lambda\sqrt{\sigma/(1-R)}] \tag{15}$$

The value t_2 can be calculated by the formula from [7]:

$$t_2 = \psi_0 \exp(-C_0\sqrt{\sigma\sqrt{L}}) \tag{16}$$

where $C_0 = \lambda f(\theta)^{1/2}(2r)^{-1/4}$ [7].

Substituting Eqs.(15),(16) into (14) one can obtain the relation between the critical density of microcracks and the size of the maximal microcracks cluster, at which the crack (i.e. the microcracks cluster) begins to grow autocatalytically:

$$L_{cr} = (\lambda^4/C_0^4)(1-R)^{-1} \tag{17}$$

Having solved the equations (11),(12) and (17) numerically, one can determine the condition of changing the regime of microcracks accumulation in the loaded body, the critical density of microcracks R_{cr} and the size of the initial crack, which is initiated due to the microcracks coalescence and begins to propagate.

As discussed above, when the crack size is smaller than the value being determined by Eq.(17), the fractal dimension of crack is about 1.89; when the crack size is greater than this value, the fractal dimension of the crack is about 1.3...1.5.

One can conclude that when the size of the loaded body is less than the critical size of the initial crack (which depends on the applied stress, i.e. this condition is determined not only by the size of the body but by the applied stress as well), the specific energy needed for the formation of unit new surface is much more than when the loaded body is greater than the value L_{cr}. It means that the scale effect can be explained not only with the use of the statistical theories of failure, but also on the basis of the fractal model of fracture : this effect is determined not only by the amount of the microdefects in a body (which is proportional to the volume of the body [21]), but by the different specific energies of fracture at the different conditions of the crack formation as well.

6 Determination of the time-to-fracture

The time-to-fracture of a loaded body can be determined by the following formula:

$$t_f = t_{f1} + t_{f2} \tag{18}$$

where t_{f1} and t_{f2} are the duration of the process of the formation of the initial crack, and the time of the crack growth up to the breakage of the body, respectively.

Substituting the critical density of microcracks R_{cr}, at which the initial crack is formed, into the Eq.(8) and integrating, one can obtain after some arrangements:

$$\begin{aligned} t_{f1}/\psi_0 &= \exp(-\gamma\sigma) - (1-R_{cr})\exp[-\gamma\sigma/(1-R_{cr})] + \\ &+\gamma\sigma[Ei(-\gamma\sigma) - (1-R_{cr})^{-1}Ei(-\gamma\sigma/(1-R_{cr}))] \end{aligned} \tag{19}$$

where Ei - the exponential integral, $Ei(x) = \int\limits_{-\infty}^{x} (1/y)e^y dy$.

The time needed to break a body with a crack can be calculated by the following formula [7]:

$$t_{f2} = [2\psi_0(C_0^2\sigma)^{-D}/a^D]\Gamma_i(2D, C_0\sqrt{\sigma\sqrt{L_{cr}}}) \tag{20}$$

where Γ_i is the incomplete gamma-function, $\Gamma_i(x,y) = \int\limits_y^{\infty} t^{x-1}e^{-t}dt$.

The equations (18) -(20) determine the time-to-fracture of the non-cracked body with allowing for the physical mechanism of the crack initiation and growth. One can see that both t_{f1} and t_{f2} decrease rapidly when the applied stress increases.

7 Computation and discussions

To study the time-to-fracture versus applied stress relation, Eqs.(11)-(19) were solved numerically.

Figure 1 shows a plot of the time to formation of the initial crack t_{f1} versus applied stress σ. The values t_{f1} and σ are normalized by the kinetic constants of the material ψ_0 and $1/\gamma$, respectively (i.e. $t'_{f1} = t_{f1}/\psi_0$, and $\sigma' = \gamma\sigma$) . Figure 2 shows a plot of the time -to-fracture t_{f2} of a body with an initial crack versus applied stress (curve 1). The values t_{f2} and σ are normalized by the $2\psi_0 C_0/a^D$, and $1/C_0$, respectively: $t''_{f2} = t_{f2}/(2\psi_0 C_0/a^D)$, and $\sigma'' = C_0\sigma$. For comparison purposes, the exponential function (1) is also presented in the Fig.2 (curve 2).

When determining the size of the initial crack and the microcracks density which corresponds to the change in the regime of the microcracks accumulation the following values were used: x = 0.3 [20], $\nu = 1.33$, $\theta = 45$ grad, r = a = 0.1, $\nu_p = 0.25$, n =10. The exponential integral was calculated with the use of the polinomial approximation [24]: $xe^x Ei(x) = (x^2 + 2.334733x + 0.250621)/(x^2 + 3.330657x + 1.681534) + \Delta$. The incomplete gamma-function was calculated numerically as well.

The value $\gamma\sigma$ was varied from 1 to 8.

It can be seen from Fig.1 and 2, that both the time to the formation of the initial crack and the time-to-fracture of the body with an initial crack decrease with increasing load. Yet, one should note that the value t_{f1} is less dependent on the stress than in the case of the exponential time-to-fracture versus stress relation. On the contrary, the dependence of t_{f2} on the applied stress which is determined by the incomplete gamma-function (see eq.(20)) is stronger than in the case of the exponential relation between t_{f2} and stress. Thus, the well-known exponential dependence of the time-to-fracture on stress (which has been developed and used by Zhurkov [1], Hsiao [2], etc) can be considered just as an averaged function which approximates the relation (18).

Then, one can compare the obtained results with results of other authors.

The conclusion about the different roughnesses of the surface of the initial part of crack and at the rest of the crack can be correlated with the data of Hellan [22], which noted as well that the surface roughness of the initial part of crack is much more than that of the rest part of the crack.

It is of interest to compare the presented model of the changes in the crack formation regimes with the analysis of two thresholds of the fatigue crack growth given by Miller [23]. Miller pointed out that there are two thresholds in the fatigue crack propagation. The first of them depends mainly on the properties of the material, and the second is determined by the stress state and the fracture mechanics principle [23]. One can see that the first threshold can correspond to the transition from the damage evolution to the crack growth (the conditions of this transition are given by Eqs.(11), (12) (17)), and the second threshold can correspond to the point of the change of character of the crack velocity versus crack length relation which has been considered in [7].

8 Conclusions

The fractal dimension and the surface roughness of the initial part of the crack are much more than these of the rest of the crack. The fractal dimension of the initial part of a crack is about 1.89; for the rest of a crack, this value is about 1.3...1.5.

The specific energy (per unit new surface) needed for the crack initiation is much more than that needed for the crack propagation.

When the size of a loaded body is less than the critical size of the initial crack at given stress, the specific energy of fracture and, consequently, the strength of the body are much more than when the body is greater than the critical size of the initial crack. It is caused just by the different fractal dimensions of the initial crack and the rest of the crack. That can furnish insights into the nature of the scale effect in fracture: the scale effect is determined not only by the amount of the microdefects in the body, but also by the different fractal dimensions (and, consequently, by the different specific energies of fracture) in formation of the initial crack and the cracks propagation.

The damage evolution law for metals has been derived on the basis of the thermofluctuational model of fracture.

The condition of changing the regime of fracture when the random initiation of the microcracks throughout the loaded body gives way to the growth of a crack is determined. The formulas for the critical microcracks density and the length of the initial crack are deduced on the basis of the percolation theory of the microcracks coalescence.

The time-to-fracture versus the load relation which takes into account the physical mechanisms of fracture is obtained.

Both the duration of the process of crack initiation from the microcracks and the time of the crack growth up to the failure of the body decrease with increasing load. The time needed for the formation of an initial crack is less dependent on the stress than in the case of the exponential dependence. The dependence of the time of the crack growth up to the failure of the body on the stress is stronger than in the case of the exponentialtial relation. Thus, the well-known Zhurkov's exponential dependence of the time-to-fracture on stress can be considered just as an averaged relation which approximates the obtained formula.

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